

# $\eta_c$ mixing effects on charmonium and $B$ meson decays

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We include the  $\eta_c$  meson into the  $\eta$ - $\eta'$ - $G$  mixing formalism constructed in our previous work, where  $G$  represents the pseudoscalar gluball. The mixing angles in this tetramixing matrix are constrained by theoretical and experimental implications from relevant hadronic processes. Especially, the angle between  $\eta_c$  and  $G$  is found to be about  $11^\circ$  from the measured decay widths of the  $\eta_c$  meson. The pseudoscalar glueball mass  $m_G$ , the pseudoscalar densities  $m_{qq,ss,cc}$  and the  $U(1)$  anomaly matrix elements associated with the mixed states are solved from the anomalous Ward identities. The solution  $m_G \approx 1.4$  GeV obtained from the  $\eta$ - $\eta'$ - $G$  mixing is confirmed, while  $m_{qq}$  grows to above the pion mass, and thus increases perturbative QCD predictions for the branching ratios  $Br(B \rightarrow \eta' K)$ . We then analyze the  $\eta_c$ -mixing effects on charmonium magnetic dipole transitions, and on the  $B \rightarrow \eta^{(\prime)} K_S$  branching ratios and CP asymmetries, which further improve the consistency between theoretical predictions and data. A predominant observation is that the  $\eta_c$  mixing enhances the perturbative QCD predictions for  $Br(B \rightarrow \eta' K)$  by 18%, but does not alter those for  $Br(B \rightarrow \eta K)$ . The puzzle due to the large  $Br(B \rightarrow \eta' K)$  data is then resolved.

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## I. INTRODUCTION

It has been known that the gluonic and charm contents of the light pseudoscalar mesons  $\eta$  and  $\eta'$  may have a crucial impact on studies of many hadronic processes, such as  $\eta^{(\prime)}$  electromagnetic (EM) transition form factors,  $\eta^{(\prime)}$  radiative decays, and charmonium and heavy-flavor decays into  $\eta^{(\prime)}$ . For a recent review on the gluonic effects in EM transitions and in weak decays of charm and beauty hadrons, see [1]. It remains a puzzle that theoretical predictions for the  $B \rightarrow \eta' K$  branching ratios are usually lower than data, even after taking into account the  $\eta$ - $\eta'$  mixing [2, 3]. Hence, it has been conjectured that the gluonic content of the  $\eta'$  meson plays a role in accommodating the large branching ratios [4–8]. A gluonic content of the  $\eta^{(\prime)}$  meson has been inferred from data of the radiative decays  $P \rightarrow \gamma V$  and  $V \rightarrow \gamma P$  [9, 10] and from the charmonium decays  $J/\psi \rightarrow VP$  [11, 12]. However, the gluonic contribution to the  $B \rightarrow \eta^{(\prime)}$  transition form factors was parametrized and tuned to fit data in the QCD factorization approach [13] and in the soft-collinear effective theory [14], so no conclusion on its importance could be drawn. This contribution was calculated explicitly in the perturbative QCD (PQCD) approach [15] using the gluonic distribution amplitudes of the  $\eta^{(\prime)}$  meson from [16] and in QCD sum rules [17], and it was found to be small.

The charm content of the  $\eta^{(\prime)}$  meson has been introduced through the  $\eta$ - $\eta'$ - $\eta_c$  mixing [18, 19]. This formalism was extended to the tetramixing among the  $\pi$ ,  $\eta$ ,  $\eta'$ , and  $\eta_c$  mesons recently [20], whose parameter set, including the mixing angles and the hadronic parameters in the light-front constituent quark model, was determined by a fit to data of relevant meson transition form factors. An intermediate question is whether this charm content affects the  $B \rightarrow \eta^{(\prime)} K$  branching ratios [21] and their CP asymmetries [22], whose measurement might reveal new physics signals. A potential deviation has been detected between the mixing-induced CP asymmetries in the tree-dominated decays  $B \rightarrow J/\psi K_S$  and in the penguin-dominated decays  $B \rightarrow \eta' K_S$ . Whether this deviation can be interpreted as a signal of new physics depends on how large the tree pollution in the latter is. Though the  $\eta^{(\prime)}$ - $\eta_c$  mixing is small [23, 24], there is lack of quantitative estimate of its effect. It is then worthwhile to examine whether the large  $B \rightarrow \eta_c K$  amplitudes are able to compensate for the tiny mixing, and to give a sizable impact on the  $B \rightarrow \eta^{(\prime)} K$  decays.

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The above subjects demand complete and precise understanding of the gluonic and charm contents in the  $\eta^{(\prime)}$  meson. We have set up the  $\eta$ - $\eta'$ - $G$  mixing in our previous work [25], where  $G$  denotes the physical pseudoscalar glueball. This mixing was implemented into the equations of motion from the anomalous Ward identities, that connect the vacuum to  $\eta$ ,  $\eta'$ , and  $G$  transition matrix elements of the divergence of axial-vector currents to the pseudoscalar densities and the  $U(1)$  anomaly. Solving these equations, the pseudoscalar glueball mass  $m_G$  was expressed in terms of phenomenological quantities such as the  $\eta$ ,  $\eta'$  masses, the decay constants, and the mixing angles. With the mixing angles measured from the  $\phi \rightarrow \gamma\eta, \gamma\eta'$  decays by KLOE [9],  $m_G \approx 1.4$  GeV has been deduced, suggesting that the  $\eta(1405)$  meson is an interesting pseudoscalar glueball candidate [25–27]. However, the solution for the pseudoscalar density  $m_{qq}$  associated with the  $\eta_q$  component of the  $\eta^{(\prime)}$  meson is still as low as its conventional value  $m_{qq} \approx m_\pi = 0.14$  GeV. It has been postulated that a larger  $m_{qq} \approx 0.2$  GeV can enhance the  $B \rightarrow \eta'$  transition form factors, and thus the  $B \rightarrow \eta'K$  branching ratios significantly [28]. Following this vein, it was pointed out that the introduction of decay constants, suppressed by the Okubo-Zweig-Iizuka (OZI) rule [29–31], into the equations from the anomalous Ward identities can increase  $m_{qq}$  [15].

We are motivated to formulate the tetramixing among  $\eta$ ,  $\eta'$ ,  $G$ , and  $\eta_c$ , and to investigate its impact on charmonium and  $B$  meson decays in this paper. The mixing with the pion is not considered here under the isospin symmetry. As a consequence of the mixing, the  $\eta_c$  meson contains a gluonic content, that modifies the QCD calculation of its decay width. The fit to the observed  $\eta_c$  decay width determines the additional mixing angle  $\phi_Q \sim 11^\circ$  between  $G$  and  $\eta_c$ . We shall explain that the gluonic content of the  $\eta_c$  meson further improves the calculations for the decay widths of the charmonium magnetic dipole transitions  $J/\psi, \psi' \rightarrow \gamma\eta_c$  in association with the unquenched mechanism proposed in [32, 33], and renders theoretical predictions in better agreement with data. Together with the other mixing angles fixed in [25], we construct the tetramixing matrix, which implies the charm content of the  $\eta'$  meson consistent with that in [20]. Implementing the tetramixing into the equations from the anomalous Ward identities, we solve for the pseudoscalar glueball mass, the pseudoscalar densities, and the  $U(1)$  anomalies. It is found that the inclusion of the  $\eta_c$  mixing does not alter our prediction for the pseudoscalar glueball mass, but increases the pseudoscalar density  $m_{qq}$  to above the pion mass, even in the absence of the OZI-suppressed decay constants.

Moreover, the charm content of the  $\eta^{(\prime)}$  meson allows the  $B \rightarrow \eta^{(\prime)}K$  decays via the  $B \rightarrow \eta_c K$  channel. Simply adopting the amplitudes evaluated in the PQCD approach at the next-to-leading-order (NLO) accuracy [34, 35], we estimate the  $\eta_c$  mixing effects on the  $B \rightarrow \eta^{(\prime)}K$  decays. It will be demonstrated that the additional tree contribution from  $B \rightarrow \eta_c K$  increases the  $B \rightarrow \eta'K$  branching ratios by about 18%, but does not change the  $B \rightarrow \eta K$  branching ratios. Combining the mechanisms from the larger  $m_{qq}$  and the charm content, the puzzle due to the large  $B \rightarrow \eta'K$  branching ratios is resolved. On the other hand, the charm content of the  $\eta^{(\prime)}$  meson has minor influences on the direct and mixing-induced CP asymmetries in the  $B \rightarrow \eta^{(\prime)}K_S$  decays. Nevertheless, we do see the modification toward accommodating the deviation between the measured mixing-induced CP asymmetries in the  $B \rightarrow J/\psi K_S$  and  $B \rightarrow \eta' K_S$  decays. The  $\eta_c$  mixing effects on the CP asymmetries in the  $B \rightarrow \eta K_S$  decays are also negligible.

We set up the  $\eta$ - $\eta'$ - $G$ - $\eta_c$  tetramixing formalism in Sec. II, which contains one more mixing angle  $\phi_Q$  between  $G$  and  $\eta_c$  compared to the  $\eta$ - $\eta'$ - $G$  mixing. The angle  $\phi_Q$  is then determined, and the resultant gluonic and charm contents of the  $\eta^{(\prime)}$  meson are compared with those obtained in the literature. In Sec. III we solve for the pseudoscalar glueball mass, the pseudoscalar densities, and the  $U(1)$  anomalies appearing in the mixing formalism, and discuss their phenomenological implications. The  $\eta_c$  mixing effects on charmonium magnetic dipole transitions and on the  $B \rightarrow \eta^{(\prime)}K$  decays are also investigated. Section IV contains the summary and comments on other works, that present observations different from ours.

## II. $\eta$ - $\eta'$ - $G$ - $\eta_c$ MIXING

In this section we formulate the  $\eta$ - $\eta'$ - $G$ - $\eta_c$  tetramixing, determine the involved mixing angles, and implement the mixing into the equations of motion from the anomalous Ward identities.

### A. Mixing matrix

We combine the Feldmann-Kroll-Stech (FKS) formalism for the  $\eta$ - $\eta'$ - $\eta_c$  mixing [18, 19] and for the  $\eta$ - $\eta'$ - $G$  mixing [25], in which the conventional singlet-octet basis and the quark-flavor basis  $q\bar{q} \equiv (u\bar{u} + d\bar{d})/\sqrt{2}$  and  $s\bar{s}$  [36], labeled by  $\eta_q$  and  $\eta_s$ , respectively, were adopted. We further introduce the unmixed glueball state  $g$  and the unmixed heavy-quark state  $\eta_Q$ . Let the matrix  $U_{34}$  ( $U_{14}$ ,  $U_{12}$ ) represent a rotation with the 3-4 (1-4, 1-2) plane fixed. It is natural

to first mix those singlet states  $\eta_1$ ,  $g$ , and  $\eta_Q$ , and then mix  $\eta_1$  and  $\eta_8$  to form the physical states  $\eta$ ,  $\eta'$ :

$$\begin{pmatrix} |\eta\rangle \\ |\eta'\rangle \\ |G\rangle \\ |\eta_c\rangle \end{pmatrix} = U_{34}(\theta)U_{14}(\phi_G)U_{12}(\phi_Q) \begin{pmatrix} |\eta_8\rangle \\ |\eta_1\rangle \\ |g\rangle \\ |\eta_Q\rangle \end{pmatrix}, \quad (1)$$

with the rotational matrices

$$U_{34}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U_{14}(\phi_G) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_G & \sin\phi_G & 0 \\ 0 & -\sin\phi_G & \cos\phi_G & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$U_{12}(\phi_Q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi_Q & \sin\phi_Q \\ 0 & 0 & -\sin\phi_Q & \cos\phi_Q \end{pmatrix}. \quad (2)$$

That is, we assume that the octet state  $\eta_8$  does not mix with the glueball, and that the heavy-flavor state mixes with the pseudoscalar glueball more dominantly than with the  $\eta_1$  state.

The octet and singlet states are related to the flavor states via

$$\begin{pmatrix} |\eta_8\rangle \\ |\eta_1\rangle \\ |g\rangle \\ |\eta_Q\rangle \end{pmatrix} = U_{34}(\theta_i) \begin{pmatrix} |\eta_q\rangle \\ |\eta_s\rangle \\ |g\rangle \\ |\eta_Q\rangle \end{pmatrix}, \quad (3)$$

where  $\theta_i$  is the ideal mixing angle with  $\cos\theta_i = \sqrt{1/3}$  and  $\sin\theta_i = \sqrt{2/3}$ , i.e.,  $\theta_i = 54.7^\circ$ . The flavor states are then transformed into the physical states through the mixing matrix

$$U(\theta, \phi_G, \phi_Q) = U_{34}(\theta)U_{14}(\phi_G)U_{12}(\phi_Q)U_{34}(\theta_i),$$

$$= \begin{pmatrix} c\theta c\theta_i - s\theta c\phi_G s\theta_i & -c\theta s\theta_i - s\theta c\phi_G c\theta_i & -s\theta s\phi_G c\phi_Q & -s\theta s\phi_G s\phi_Q \\ s\theta c\theta_i + c\theta c\phi_G s\theta_i & -s\theta s\theta_i + c\theta c\phi_G c\theta_i & c\theta s\phi_G c\phi_Q & c\theta s\phi_G s\phi_Q \\ -s\phi_G s\theta_i & -s\phi_G c\theta_i & c\phi_G c\phi_Q & c\phi_G s\phi_Q \\ 0 & 0 & -s\phi_Q & c\phi_Q \end{pmatrix}, \quad (4)$$

with the notations  $c\theta \equiv \cos\theta$  and  $s\theta \equiv \sin\theta$ . Equation (4) approaches the mixing matrix in [25] in the  $\phi_Q \rightarrow 0$  limit, and the  $\eta$ - $\eta'$  mixing matrix [18, 19] in the  $\phi_Q, \phi_G \rightarrow 0$  limit.

As stated in the Introduction, we have assumed isospin symmetry, i.e. no mixing with  $\pi^0$ , and ignored other possible admixtures from radial excitations. The decay constants  $f_q$ ,  $f_s$  and  $f_c$  are defined via the matrix elements

$$\begin{aligned} \langle 0 | \bar{q} \gamma^\mu \gamma_5 q | \eta_q(P) \rangle &= -\frac{i}{\sqrt{2}} f_q P^\mu, \\ \langle 0 | \bar{s} \gamma^\mu \gamma_5 s | \eta_s(P) \rangle &= -i f_s P^\mu, \\ \langle 0 | \bar{c} \gamma^\mu \gamma_5 c | \eta_Q(P) \rangle &= -i f_c P^\mu, \end{aligned} \quad (5)$$

for the light quark  $q = u$  or  $d$ . The other decay constants of the  $\eta_q$ ,  $\eta_s$ , and  $\eta_Q$  mesons and of the unmixed glueball, which are suppressed by the OZI rule, can be introduced in a similar way [15]:

$$\begin{aligned} \langle 0 | \bar{q} \gamma^\mu \gamma_5 q | \eta_s(P), g(P), \eta_Q(P) \rangle &= -\frac{i}{\sqrt{2}} f_{s,g,c}^q P^\mu, \\ \langle 0 | \bar{s} \gamma^\mu \gamma_5 s | \eta_q(P), g(P), \eta_Q(P) \rangle &= -i f_{q,g,c}^s P^\mu, \\ \langle 0 | \bar{c} \gamma^\mu \gamma_5 c | \eta_q(P), \eta_s(P), g(P) \rangle &= -i f_{q,s,g}^c P^\mu. \end{aligned} \quad (6)$$

The decay constants associated with the  $\eta$ ,  $\eta'$ ,  $G$ , and  $\eta_c$  physical states in the following matrix elements

$$\begin{aligned} \langle 0 | \bar{q} \gamma^\mu \gamma_5 q | \eta(P), \eta'(P), G(P), \eta_c(P) \rangle &= -\frac{i}{\sqrt{2}} f_{\eta,\eta',G,\eta_c}^q P^\mu, \\ \langle 0 | \bar{s} \gamma^\mu \gamma_5 s | \eta(P), \eta'(P), G(P), \eta_c(P) \rangle &= -i f_{\eta,\eta',G,\eta_c}^s P^\mu, \\ \langle 0 | \bar{c} \gamma^\mu \gamma_5 c | \eta(P), \eta'(P), G(P), \eta_c(P) \rangle &= -i f_{\eta,\eta',G,\eta_c}^c P^\mu, \end{aligned} \quad (7)$$

are related to those associated with the  $\eta_q$ ,  $\eta_s$ ,  $g$ ,  $\eta_Q$  states through

$$\begin{pmatrix} f_{\eta}^q & f_{\eta}^s & f_{\eta}^c \\ f_{\eta'}^q & f_{\eta'}^s & f_{\eta'}^c \\ f_G^q & f_G^s & f_G^c \\ f_{\eta_c}^q & f_{\eta_c}^s & f_{\eta_c}^c \end{pmatrix} = U(\theta, \phi_G, \phi_Q) \begin{pmatrix} f_q & f_q^s & f_q^c \\ f_s^q & f_s^s & f_s^c \\ f_g^q & f_g^s & f_g^c \\ f_c^q & f_c^s & f_c^c \end{pmatrix}. \quad (8)$$

We sandwich the equations of motion for the anomalous Ward identities

$$\begin{aligned} \partial_\mu(\bar{q}\gamma^\mu\gamma_5 q) &= 2im_q \bar{q}\gamma_5 q + \frac{\alpha_s}{4\pi} G_{\mu\nu} \tilde{G}^{\mu\nu}, \\ \partial_\mu(\bar{s}\gamma^\mu\gamma_5 s) &= 2im_s \bar{s}\gamma_5 s + \frac{\alpha_s}{4\pi} G_{\mu\nu} \tilde{G}^{\mu\nu}, \\ \partial_\mu(\bar{c}\gamma^\mu\gamma_5 c) &= 2im_c \bar{c}\gamma_5 c + \frac{\alpha_s}{4\pi} G_{\mu\nu} \tilde{G}^{\mu\nu}, \end{aligned} \quad (9)$$

between vacuum and  $|\eta\rangle$ ,  $|\eta'\rangle$ ,  $|G\rangle$ , and  $|\eta_c\rangle$ , where  $m_{q,s,c}$  are the quark masses,  $G_{\mu\nu}$  is the field-strength tensor and  $\tilde{G}^{\mu\nu}$  is the dual field-strength tensor. Following the procedure in [15], we derive

$$M_{qsgc} = U^\dagger(\theta, \phi_G, \phi_Q) M^2 U(\theta, \phi_G, \phi_Q) \tilde{J}, \quad (10)$$

in which the matrices are written as

$$M_{qsgc} = \begin{pmatrix} m_{qq}^2 + \sqrt{2}G_q/f_q & m_{sq}^2 + G_q/f_s & m_{cq}^2 + G_q/f_c \\ m_{qs}^2 + \sqrt{2}G_s/f_q & m_{ss}^2 + G_s/f_s & m_{cs}^2 + G_s/f_c \\ m_{qg}^2 + \sqrt{2}G_g/f_q & m_{sg}^2 + G_g/f_s & m_{cg}^2 + G_g/f_c \\ m_{qc}^2 + \sqrt{2}G_c/f_q & m_{sc}^2 + G_c/f_s & m_{cc}^2 + G_c/f_c \end{pmatrix}, \quad (11)$$

$$M^2 = \begin{pmatrix} m_\eta^2 & 0 & 0 & 0 \\ 0 & m_{\eta'}^2 & 0 & 0 \\ 0 & 0 & m_G^2 & 0 \\ 0 & 0 & 0 & m_{\eta_c}^2 \end{pmatrix}, \quad \tilde{J} = \begin{pmatrix} 1 & f_q^s/f_s & f_q^c/f_c \\ f_s^q/f_q & 1 & f_s^c/f_c \\ f_g^q/f_q & f_g^s/f_s & f_g^c/f_c \\ f_c^q/f_q & f_c^s/f_s & 1 \end{pmatrix}, \quad (12)$$

with the  $\eta$ ,  $\eta'$ ,  $G$ ,  $\eta_c$  meson masses  $m_{\eta,\eta',G,\eta_c}$ , and the abbreviations for the pseudoscalar densities and the  $U(1)$  anomaly matrix elements

$$\begin{aligned} m_{qq,qs,qg,qc}^2 &\equiv \frac{\sqrt{2}}{f_q} \langle 0 | m_u \bar{u} i \gamma_5 u + m_d \bar{d} i \gamma_5 d | \eta_q, \eta_s, g, \eta_Q \rangle, \\ m_{sq,ss,sg,sc}^2 &\equiv \frac{2}{f_s} \langle 0 | m_s \bar{s} i \gamma_5 s | \eta_q, \eta_s, g, \eta_Q \rangle, \\ m_{cq,cs,cg,cc}^2 &\equiv \frac{2}{f_c} \langle 0 | m_c \bar{c} i \gamma_5 c | \eta_q, \eta_s, g, \eta_Q \rangle, \\ G_{q,s,g,c} &\equiv \langle 0 | \alpha_s G \tilde{G} / (4\pi) | \eta_q, \eta_s, g, \eta_Q \rangle. \end{aligned} \quad (13)$$

In the following analysis we neglect all the OZI-suppressed decay constants defined in Eq. (6) according to the FKS

scheme. Equation (12) then leads to the following equations explicitly

$$M_{qsgc}^{11} = m_\eta^2(c\theta c\theta_i - s\theta c\phi_G s\theta_i)^2 + m_{\eta'}^2(s\theta c\theta_i + c\theta c\phi_G s\theta_i)^2 + m_G^2(s\phi_G s\theta_i)^2, \quad (14)$$

$$M_{qsgc}^{12} = -m_\eta^2(c\theta c\theta_i - s\theta c\phi_G s\theta_i)(c\theta s\theta_i + s\theta c\phi_G c\theta_i) + m_{\eta'}^2(s\theta c\theta_i + c\theta c\phi_G s\theta_i)(-s\theta s\theta_i + c\theta c\phi_G c\theta_i) + m_G^2(s\phi_G)^2 c\theta_i s\theta_i, \quad (15)$$

$$M_{qsgc}^{13} = -m_\eta^2(c\theta c\theta_i - s\theta c\phi_G s\theta_i)s\theta s\phi_G s\phi_Q + m_{\eta'}^2(s\theta c\theta_i + c\theta c\phi_G s\theta_i)c\theta s\phi_G s\phi_Q - m_G^2 c\phi_G s\phi_G s\theta_i s\phi_Q, \quad (16)$$

$$M_{qsgc}^{21} = M_{qsgc}^{12}, \quad (17)$$

$$M_{qsgc}^{22} = m_\eta^2(c\theta s\theta_i + s\theta c\phi_G c\theta_i)^2 + m_{\eta'}^2(-s\theta s\theta_i + c\theta c\phi_G c\theta_i)^2 + m_G^2(s\phi_G c\theta_i)^2, \quad (18)$$

$$M_{qsgc}^{23} = m_\eta^2(c\theta s\theta_i + s\theta c\phi_G c\theta_i)s\theta s\phi_G s\phi_Q + m_{\eta'}^2(-s\theta s\theta_i + c\theta c\phi_G c\theta_i)c\theta s\phi_G s\phi_Q - m_G^2 c\phi_G s\phi_G c\theta_i s\phi_Q, \quad (19)$$

$$M_{qsgc}^{31} = -m_\eta^2(c\theta c\theta_i - s\theta c\phi_G s\theta_i)s\theta s\phi_G c\phi_Q + m_{\eta'}^2(s\theta c\theta_i + c\theta c\phi_G s\theta_i)c\theta s\phi_G c\phi_Q - m_G^2 c\phi_G s\phi_G s\theta_i c\phi_Q, \quad (20)$$

$$M_{qsgc}^{32} = m_\eta^2(c\theta s\theta_i + s\theta c\phi_G c\theta_i)s\theta s\phi_G c\phi_Q + m_{\eta'}^2(-s\theta s\theta_i + c\theta c\phi_G c\theta_i)c\theta s\phi_G c\phi_Q - m_G^2 c\phi_G s\phi_G c\theta_i c\phi_Q, \quad (21)$$

$$M_{qsgc}^{33} = m_\eta^2(s\theta s\phi_G)^2 c\phi_Q s\phi_Q + m_{\eta'}^2(c\theta s\phi_G)^2 c\phi_Q s\phi_Q + m_G^2(c\phi_G)^2 c\phi_Q s\phi_Q - m_{\eta_c}^2 c\phi_Q s\phi_Q, \quad (22)$$

$$M_{qsgc}^{41} = -m_\eta^2(c\theta c\theta_i - s\theta c\phi_G s\theta_i)s\theta s\phi_G s\phi_Q + m_{\eta'}^2(s\theta c\theta_i + c\theta c\phi_G s\theta_i)c\theta s\phi_G s\phi_Q - m_G^2 c\phi_G s\phi_G s\theta_i s\phi_Q, \quad (23)$$

$$M_{qsgc}^{42} = m_\eta^2(c\theta s\theta_i + s\theta c\phi_G c\theta_i)s\theta s\phi_G s\phi_Q + m_{\eta'}^2(-s\theta s\theta_i + c\theta c\phi_G c\theta_i)c\theta s\phi_G s\phi_Q - m_G^2 c\phi_G s\phi_G c\theta_i s\phi_Q, \quad (24)$$

$$M_{qsgc}^{43} = m_\eta^2(s\theta s\phi_G s\phi_Q)^2 + m_{\eta'}^2(c\theta s\phi_G s\phi_Q)^2 + m_G^2(c\phi_G s\phi_Q)^2 + m_{\eta_c}^2(c\phi_Q)^2. \quad (25)$$

## B. Mixing angles

There is already extensive discussion on the determination of the mixing angle  $\theta$  in the literature, whose value still varies in a finite range. For example,  $-17^\circ < \theta < -11^\circ$  has been extracted in [37], assuming the presence of the gluonic content in the  $\eta'$  meson. We choose  $\theta = -11^\circ$ , which corresponds to a sizable gluonic content [37], and is appropriate for our choice of  $\phi_G$  below. The value of  $\phi_G$  varies in a wide range, depending on the parametrization of the mixing matrix, experimental inputs, and fitting procedures [25]. Even its central value can take a number between  $10^\circ \lesssim \phi_G \lesssim 30^\circ$ , such as  $\phi_G = (12 \pm 13)^\circ$  in [10] and  $\phi_G = (33 \pm 13)^\circ$  in [12]. We take the value  $\phi_G = 12^\circ$ , which was also considered in [25].

The last angle  $\phi_Q$  can be determined by the  $\eta_c$  total width and the  $\eta_c \rightarrow \gamma\gamma$  decay width. As a  $c\bar{c}(^1S_0)$  state, the value of  $\Gamma_{\text{tot}}$  is quite large among the charmonia below the  $D\bar{D}$  threshold. During the past few years, the experimental results for the  $\eta_c$  total width and the two-photon decay branching ratios vary drastically. The Particle Data Group (PDG) 2008 [38] listed  $\Gamma_{\text{tot}} = 26.7 \pm 3.0$  MeV and  $Br(\eta_c \rightarrow \gamma\gamma) = (2.4_{-0.9}^{+1.1}) \times 10^{-4}$ , respectively. In contrast, the PDG2010 [39] presents  $\Gamma_{\text{tot}} = 28.6 \pm 2.2$  MeV and  $Br(\eta_c \rightarrow \gamma\gamma) = (6.3 \pm 2.9) \times 10^{-5}$ . The BESIII Collaboration measured the  $\eta_c$  total width recently in  $\psi' \rightarrow \gamma\eta_c$  and found  $\Gamma_{\text{tot}} = 32.0 \pm 1.2 \pm 1.0$  MeV [40]. The interesting tendency is that the total width of  $\eta_c$  becomes broader than the previous measurements. Such a change favors the scenario that the  $\eta_c$  has a glueball content in association with the  $c\bar{c}(^1S_0)$  component as analyzed below. One also notices the much smaller  $Br(\eta_c \rightarrow \gamma\gamma)$  listed in the PDG2010 than in PDG2008, which may be caused by the old total width data employed in the extraction of the two-photon branching ratio. Since the averaged partial decay widths for  $\eta_c \rightarrow \gamma\gamma$  are unchanged in PDG2008 and PDG2010, i.e.  $\Gamma(\eta_c \rightarrow \gamma\gamma) = 6.7_{-0.8}^{+0.9}$  keV, it might be more appropriate to adopt the experimental data for the two-photon partial width instead of for the branching ratio in the following analysis.

The strong decay of  $c\bar{c}(^1S_0)$  annihilation via two-gluon radiation can be related to its two-photon decay. To lowest order, one has

$$\frac{\Gamma(^1S_0 \rightarrow gg)}{\Gamma(^1S_0 \rightarrow \gamma\gamma)} \simeq \frac{2\alpha_s^2}{9e_c^4\alpha_e^2} = \frac{9}{8} \left( \frac{\alpha_s}{\alpha_e} \right)^2, \quad (26)$$

where  $e_c = 2/3$  is the charge of the  $c$  quark, and  $\alpha_s$  and  $\alpha_e$  are the strong and EM coupling constants, respectively. With  $\Gamma(^1S_0 \rightarrow gg) \simeq \Gamma_{\text{tot}} = 28.6 \pm 2.2$  MeV from PDG2010 [39] and  $\Gamma(^1S_0 \rightarrow \gamma\gamma) = 6.7_{-0.8}^{+0.9}$  keV, we extract  $\alpha_s \simeq 0.41 \sim 0.49$  ( $\alpha_s \simeq 0.72 \sim 1.2$  if one adopts the branching ratio  $Br(\eta_c \rightarrow \gamma\gamma) = (6.3 \pm 2.9) \times 10^{-5}$  from PDG2010 [39]), which is much larger than the running coupling  $\alpha_s(m_c) \simeq 0.24 \sim 0.26$  [39]. Even with the first-order QCD correction [41] taken into account:

$$\frac{\Gamma(^1S_0 \rightarrow gg)}{\Gamma(^1S_0 \rightarrow \gamma\gamma)} \simeq \frac{9}{8} \left( \frac{\alpha_s}{\alpha_e} \right)^2 \frac{(1 + 4.8 \frac{\alpha_s}{\pi})}{(1 - 3.4 \frac{\alpha_s}{\pi})}, \quad (27)$$

we still have larger values  $\alpha_s \simeq 0.28 \sim 0.33$ .

As defined in Eq. (4), the  $\eta_c$  meson has the wave function

$$|\eta_c\rangle = -s\phi_Q|g\rangle + c\phi_Q|\eta_Q\rangle. \quad (28)$$

The strong decay amplitude can be parametrized as

$$\begin{aligned} \langle gg|\hat{V}|\eta_c\rangle &= -s\phi_Q\langle gg|\hat{V}|g\rangle + c\phi_Q\langle gg|\hat{V}|\eta_Q\rangle \\ &= \left( -\frac{\pi s\phi_Q}{\alpha_s} + c\phi_Q \right) \langle gg|\hat{V}|\eta_Q\rangle, \end{aligned} \quad (29)$$

where  $\hat{V}$  denotes the potential for annihilating the  $c\bar{c}$  states and the glueball into two gluons. Since the glueball does not pay a price for coupling to the  $|gg\rangle$  state, we have the ratio  $\langle gg|\hat{V}|g\rangle/\langle gg|\hat{V}|\eta_Q\rangle = \pi/\alpha_s$  given by the empirical gluon power counting [42]. Therefore, the mixing provides a correction to the strong decays different from the correction to the EM coupling  $\alpha_e^2 \rightarrow (c\phi_Q)^2\alpha_e^2$ . Consequently, one has, to the leading QCD correction,

$$\frac{\Gamma(\eta_c \rightarrow gg)}{\Gamma(\eta_c \rightarrow \gamma\gamma)} = \frac{9}{8} \frac{(-\pi t\phi_Q + \alpha_s)^2}{\alpha_e^2} \frac{(1 + 4.8 \frac{\alpha_s}{\pi})}{(1 - 3.4 \frac{\alpha_s}{\pi})}, \quad (30)$$

with  $t\phi_Q \equiv \tan\phi_Q$ . Requiring  $\alpha_s = \alpha_s(m_c) \simeq 0.25$ , we deduce  $\phi_Q \simeq -1.4^\circ$  or  $10.4^\circ$  ( $\phi_Q \simeq -6.9^\circ$  or  $15.6^\circ$ , if  $Br(\eta_c \rightarrow \gamma\gamma) = (6.3 \pm 2.9) \times 10^{-5}$  is adopted). If inserting the recent BESIII value  $\Gamma_{\text{tot}} = 32.0 \pm 1.2 \pm 1.0$  MeV [40], we extract  $\phi_Q \simeq -1.7^\circ$  or  $10.8^\circ$ . It is noted in advance that the data of the radiative decays  $J/\psi, \psi' \rightarrow \gamma\eta_c$  favor the positive solution as illustrated in Sec. III C. Based on the above determinations of  $\phi_Q$ , we adopt  $\phi_Q = 11^\circ$  in this work for the purpose of estimation.

We should mention a recent calculation of the  $^1S_0$  decays into light hadrons and two photons in the framework of nonrelativistic QCD (NRQCD) [43], in which the  $O(\alpha_s v^2)$  corrections were found to be crucial for accommodating the observed  $\eta_c \rightarrow gg$  and  $\gamma\gamma$  widths. The involved long-distance matrix elements for the  $\eta_c$  meson still suffer large uncertainties, and their values were obtained by data fitting in [43]. Hence, it is a fair comment that the present data cannot distinguish the NRQCD results with the  $O(\alpha_s v^2)$  corrections from the mixing scenario with a small glueball component in the  $\eta_c$  meson. Meanwhile, it should be realized that the presence of the glueball component would affect the total and EM decay widths of the  $\eta_c$  meson differently, and thus make an impact on their ratio at leading order.

With the angles  $\theta = -11^\circ$ ,  $\phi_G = 12^\circ$  and  $\phi_Q = 11^\circ$ , the mixing matrix in Eq. (4) is explicitly written as

$$U = \begin{pmatrix} 0.720 & -0.693 & 0.039 & 0.008 \\ 0.673 & 0.711 & 0.200 & 0.039 \\ -0.170 & -0.120 & 0.960 & 0.186 \\ 0 & 0 & -0.191 & 0.982 \end{pmatrix}. \quad (31)$$

Compared to the parametrization for the  $\eta$ - $\eta'$ - $\eta_c$  mixing [18, 19], the role of the small angle  $\theta_c = -1^\circ \pm 0.1^\circ$  has been played by our  $\phi_G\phi_Q$ . The matrix for the  $\eta$ - $\eta'$ - $\eta_c$  mixing is given by [18]

$$\begin{aligned} |\eta\rangle &= 0.77|\eta_q\rangle - 0.63|\eta_s\rangle - 0.006|\eta_Q\rangle, \\ |\eta'\rangle &= 0.63|\eta_q\rangle + 0.77|\eta_s\rangle - 0.016|\eta_Q\rangle, \\ |\eta_c\rangle &= 0.015|\eta_q\rangle + 0.008|\eta_s\rangle + |\eta_Q\rangle, \end{aligned} \quad (32)$$

in which the charm content in the  $\eta^{(\prime)}$  meson has a sign opposite to that in Eq. (31). The fit to the data of  $\Gamma(J/\psi \rightarrow \gamma\eta')/\Gamma(J/\psi \rightarrow \gamma\eta_c)$  [18] actually cannot discriminate the sign of these coefficients. Another difference appears in the  $\eta_{q,s}$  components of the  $\eta_c$  meson. Since these are small components, there is no inconsistency between the results in Eqs. (31) and (32).

The charm contents of the  $\eta$  and  $\eta'$  mesons have the same sign in the FKS scheme. This feature differs from the parametrization for the  $\pi$ - $\eta$ - $\eta'$ - $\eta_c$  tetramixing based on the group decomposition  $SO(4) = SO(3) \otimes SO(3)$  [20],

$$U_{\pi\eta\eta'\eta_c} = \begin{pmatrix} 0.9895 & 0.0552 & -0.1119 & 0.0342 \\ -0.1082 & 0.8175 & -0.5614 & -0.0259 \\ 0.0590 & 0.5696 & 0.8160 & 0.0452 \\ -0.0395 & -0.0065 & -0.0478 & 0.9960 \end{pmatrix}, \quad (33)$$

where the charm contents of the  $\eta$  and  $\eta'$  mesons are opposite in sign. A careful look reveals that these matrix elements are small due to the destruction of large numbers, so they are sensitive to experimental inputs. Varying the inputs slightly, one could get the matrix elements of the same sign in [20]. The  $\eta$  meson has a pion component  $-0.11$ , while the  $\eta'$  meson has a smaller pion component  $0.06$  as indicated in Eq. (33). Therefore, it is appropriate to compare the charm contents of the  $\eta'$  meson in Eqs. (31) and (33), both of which are about  $0.04$ .

### III. $\eta_c$ -MIXING EFFECTS

In this section we solve Eq. (10) from the anomalous Ward identities, and then investigate the phenomenological impacts from the  $\eta_c$  mixing on charmonium magnetic dipole transitions and on the  $B \rightarrow \eta^{(\prime)} K$  decays.

#### A. Decay constants

To obtain the decay constants defined in Eq. (8), we need the inputs of  $f_q$ ,  $f_s$  and  $f_c$  for the flavor eigenstates. The value of  $f_q$  is close to the pion decay constant  $f_\pi$ , such as  $f_q = (1.07 \pm 0.02)f_\pi$  extracted in [18]. Our analysis indicates that the variation of  $f_q$  has almost no influence, so we simply set it to  $f_q = f_\pi = 131$  MeV. The value of  $f_s$  is more uncertain, for which the result  $f_s/f_q \approx 1.2 \sim 1.3$  also from [18] is employed. Since the ratio  $f_s/f_q$  gives a more significant effect, we shall examine how our outcomes depend on its variation. For  $f_c$ , we adopt the value  $f_c = 487.4$  MeV [20]. Ignoring the terms suppressed by the OZI rule, Eq. (8) leads to

$$\begin{pmatrix} f_\eta^q & f_\eta^s & f_\eta^c \\ f_{\eta'}^q & f_{\eta'}^s & f_{\eta'}^c \\ f_G^q & f_G^s & f_G^c \\ f_{\eta_c}^q & f_{\eta_c}^s & f_{\eta_c}^c \end{pmatrix} = U \begin{pmatrix} f_q & 0 & 0 \\ 0 & f_s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f_c \end{pmatrix} = \begin{pmatrix} 113 & -90.8 & 3.69 \\ 106 & 93.1 & 19.0 \\ -26.7 & -15.7 & 91.0 \\ 0 & 0 & 478 \end{pmatrix}. \quad (34)$$

The larger decay constants  $f_{\eta^{(\prime)}}^{q,s}$  are close to those appearing in the literature [18], and in the perturbative calculations of the  $B \rightarrow \eta^{(\prime)} K$  decays [13, 34, 44]. It is natural that  $f_{\eta_c}^c$  is almost the same as  $f_c$ . Those decay constants  $f_G^{q,s,c}$  may have phenomenological applications, after the pseudoscalar glueball is identified. Because of  $\theta = -11^\circ$ , we have the ratio  $f_\eta^c/f_{\eta'}^c = -\tan\theta = 0.24$ , which is similar to  $-0.006/(-0.016) = 0.38$  in [18, 19], but has an opposite sign to  $-0.0259/0.0452 = -0.57$  in [20]. Note that the sign of  $f_{\eta'}^c$  is still not certain, which was found to be positive in [21, 45], but negative in [18, 19]. According to [46], a positive  $f_{\eta^{(\prime)}}^c$  would enhance the  $\eta^{(\prime)}$  transition form factor in the light-cone PQCD approach. Nevertheless, the enhancement will be compensated by varying other parameters, such as the constituent quark masses, so that the corresponding data can still be accommodated in their theoretical framework [46]. Our result  $f_{\eta'}^c = 19.0$  MeV, arising from the phenomenological determination, is approximately equal to  $21.9$  MeV in [20], and lower than  $50$ - $180$  MeV computed from QCD low energy theorem [21]. The similarity to the value in [20] is nontrivial, since we have constructed the mixing matrix via the angles  $\theta$ ,  $\phi_G$  and  $\phi_Q$ , which were determined in a different way. It has been conjectured [47] that the value of  $f_{\eta'}^c$  was overestimated in [21]. Hence, we disagree with their speculation that the charm content of the  $\eta'$  meson alone can exhaust the large  $B \rightarrow \eta' K$  branching ratios.

Our result  $f_{\eta'}^c = 19.0$  MeV is larger than  $-(6.3 \pm 0.6)$  MeV in magnitude in [18], and about 8 times larger than  $2.4$  MeV extracted from the data of  $Br(J/\psi \rightarrow \gamma\eta')/Br(J/\psi \rightarrow \gamma\eta_c)$  [24]. As stated above, the data of  $Br(J/\psi \rightarrow \gamma\eta')/Br(J/\psi \rightarrow \gamma\eta_c)$  cannot fix the sign of  $f_{\eta'}^c$  in [18]. Besides, the smaller value in [18] is attributed to the tiny  $\eta_c$  mixing, which is a consequence of the anomalous Ward identities in Eq. (9) without the OZI-suppressed terms. Including the OZI-suppressed terms, a larger range for  $f_{\eta'}^c$  is expected. The analysis in [18, 24], together with that in [48, 49] which also concluded a small  $\eta'$ - $\eta_c$  mixing, were performed in the  $\eta$ - $\eta'$ - $\eta_c$  mixing formalism. That is, only a single channel  $J/\psi \rightarrow \gamma\eta_Q \rightarrow \gamma\eta'$  contributes to  $Br(J/\psi \rightarrow \gamma\eta')$ , which is perhaps too strong of an assumption. With the more general  $\eta$ - $\eta'$ - $G$ - $\eta_c$  tetramixing, an additional channel  $J/\psi \rightarrow \gamma G \rightarrow \gamma\eta'$  exists. For

the purpose of illustration, we regard  $\eta(1405)$  as the pseudoscalar glueball [25], and assign a finite fraction of the  $J/\psi \rightarrow \gamma\eta(1405/1475)$  amplitude to  $J/\psi \rightarrow \gamma\eta(1405)$ . Employing the data in PDG2010 [39], it is easy to find that the above two channels, multiplied by the mixing matrix elements 0.039 and 0.2 in Eq. (31), respectively, are comparable to each other. If a destructive interference occurs between the two channels, a larger  $\eta'-\eta_c$  mixing will be allowed. On the other hand, some theoretical estimates on  $f_{\eta'}^c$  also implied a small  $\eta'-\eta_c$  mixing [13, 23, 47], in which the annihilation of the  $c\bar{c}$  pair into two gluons, followed by their combination into the  $\eta'$  meson, was considered. Strictly speaking, what they investigated is the extrinsic charm content of the  $\eta'$  meson, while the  $\eta'-\eta_c$  mixing results in the intrinsic charm content [20]. Therefore, their observation does not contradict to ours, and supports that the intrinsic charm content analyzed in this work might be more relevant.

### B. Glueball mass, Pseudoscalar densities, and $U(1)$ anomalies

Given the decay constants  $f_q$ ,  $f_s$  and  $f_c$ , we solve Eq. (10) in order to get an idea of the magnitude of the pseudoscalar glueball mass, the pseudoscalar densities, and the  $U(1)$  anomaly matrix elements. There are 12 equations from the  $4 \times 3$  matrix equation in Eq. (10). Even having dropped all the OZI-suppressed decay constants, there are still too many unknowns listed above. Hence, we intend to drop the OZI-suppressed pseudoscalar densities, except  $m_{cq}^2$ ,  $m_{cs}^2$  and  $m_{cg}^2$ , whose values can serve as a check of their scaling behavior in the large  $N_c$  limit [25],  $N_c$  being the number of colors. Then one of Eqs. (20), (21), (23), and (24) becomes redundant, since both the ratio of Eq. (20) over Eq. (21), and the ratio of Eq. (23) over Eq. (24) give

$$\frac{-m_{\eta'}^2(c\theta c\theta_i - s\theta c\phi_G s\theta_i)s\theta + m_{\eta'}^2(s\theta c\theta_i + c\theta c\phi_G s\theta_i)c\theta - m_G^2 c\phi_G s\theta_i}{m_{\eta'}^2(c\theta s\theta_i + s\theta c\phi_G c\theta_i)s\theta + m_{\eta'}^2(-s\theta s\theta_i + c\theta c\phi_G c\theta_i)c\theta - m_G^2 c\phi_G c\theta_i} = \frac{\sqrt{2}f_s}{f_q}, \quad (35)$$

which is identical to the formula derived in [25]. Eventually, we solve for 11 unknowns, which include the pseudoscalar glueball mass  $m_G$ , 6 pseudoscalar densities  $m_{qq}^2$ ,  $m_{ss}^2$ ,  $m_{cq}^2$ ,  $m_{cs}^2$ ,  $m_{cg}^2$ , and  $m_{cc}^2$ , and 4  $U(1)$  anomaly matrix elements  $G_q$ ,  $G_s$ ,  $G_g$ , and  $G_c$ .

According to Eq. (14),  $m_{qq}^2$  is expressed as the difference between the right-hand side of Eq. (14) and the anomaly matrix element  $G_q/f_q$ . Equation (15) implies that  $G_q$  is proportional to  $f_s$ , so the above difference strongly depends on the ratio  $f_s/f_q$  [15]. Equation (35) shows that the inclusion of the  $\eta_c$  mixing does not affect much the solution  $m_G \approx 1.4$  GeV [25]. The value of  $m_G$  is insensitive to the uncertain mixing angle  $\phi_G$  also, because of  $c\phi_G \approx 1$  for a small  $\phi_G$ . Therefore,  $m_G$  is only sensitive to the ratio  $f_s/f_q$ , and we shall consider the variation of this ratio below. The solutions corresponding to  $f_s/f_q = 1.2$  and 1.3 are collected in Table I. It is seen that only  $m_G$  and  $m_{qq}^2$  depend on the ratio  $f_s/f_q$ , and other quantities are relatively stable. The results of  $m_G$ ,  $m_{ss}^2$ ,  $G_q$ ,  $G_s$ , and  $G_g$  are similar to those from the  $\eta$ - $\eta'$ - $G$  mixing formalism [25]. Namely,  $m_{ss}^2$  in Table I respects the leading  $N_c$  relation  $m_{ss}^2 = 2m_K^2 - m_\pi^2$ , and we do not observe the enhancement claimed in [50]. Note that a larger pseudoscalar glueball mass ( $> 2$  GeV) has been postulated in a dynamical analysis of the mixing in the pseudoscalar channel [51]. However, if their assumption on the meson couplings is relaxed, a lower mass can be attained. Our solutions respect the hierarchy  $|m_{cc}^2| \gg |m_{cg}^2| \gg |m_{cq,cs}^2|$  in the large  $N_c$  limit. The values of  $m_{cc}^2$  are consistent with the relation  $m_{cc}^2 \approx m_{\eta_c}^2$  [18], but the magnitude of  $G_c$  is a bit smaller than that in [18].

It has been shown that the  $B \rightarrow \eta'$  transition form factors and the  $B \rightarrow \eta'K$  branching ratios are sensitive to the pseudoscalar density  $m_{qq}$  [28], which defines the normalization of the two-parton twist-3 distribution amplitudes for the  $\eta_q$  state. Its value is usually assumed to be the pion mass,  $m_{qq} \approx m_\pi$ , with which the branching ratios  $Br(B^\pm \rightarrow \eta'K^\pm) \approx 51 \times 10^{-6}$  and  $Br(B^0 \rightarrow \eta'K^0) \approx 50 \times 10^{-6}$  have been obtained in NLO PQCD [34]. These results are still lower than the data  $Br(B^\pm \rightarrow \eta'K^\pm) \approx (71.1 \pm 2.6) \times 10^{-6}$  and  $Br(B^0 \rightarrow \eta'K^0) \approx (66.1 \pm 3.1) \times 10^{-6}$  [52]. It was then demonstrated that  $m_{qq}$  can be increased up to 0.2 GeV by introducing the OZI-suppressed decay constants  $f_q^s$  and  $f_s^q$  into the equation for the  $\eta$ - $\eta'$  mixing [15]. With the enhanced  $m_{qq}$ , it is likely to explain the large  $Br(B \rightarrow \eta'K)$  data with larger  $B \rightarrow \eta'$  transition form factors. Since  $f_q^s$  and  $f_s^q$  are free parameters, whether  $m_{qq}$  can reach 0.2 GeV is not conclusive. Table I indicates that the  $\eta_c$  mixing, which receives a phenomenological support as elucidated in Sec. II B, can enlarge  $m_{qq}^2$  for  $f_s/f_q = 1.3$  by a factor 2, from  $m_{qq}^2 \approx 0.012$  GeV<sup>2</sup> in Eq. (29) of [25] to 0.023 GeV<sup>2</sup>. Note that the values of the mixing angle  $\theta$  for Eq. (29) in [25] and for Table I are different. However,  $m_{qq}^2$  does not much depend on the variation of  $\theta$ .

### C. Charmonium M1 transitions

There has been a long-standing puzzle from the magnetic dipole (M1) transition  $J/\psi \rightarrow \gamma\eta_c$ . In contrast to the success of the nonrelativistic potential models in offering an overall good description of the charmonium spectrum,



TABLE I: Solutions corresponding to  $f_s/f_q = 1.2$  and  $f_s/f_q = 1.3$ .

$f_q$	$f_s/f_q$	$m_G$ (GeV)	$m_{qq}^2$	$m_{ss}^2$	$m_{cq}^2$	$m_{cs}^2$	$m_{c\bar{q}}^2$	$m_{c\bar{c}}^2$ (GeV <sup>2</sup> )	$G_q$	$G_s$	$G_g$	$G_c$ (GeV <sup>3</sup> )
$f_\pi$	1.2	1.519	0.067	0.443	-0.156	-0.092	-1.197	8.648	0.053	0.031	-0.023	-0.004
$f_\pi$	1.3	1.376	0.023	0.457	-0.149	-0.081	-1.283	8.631	0.056	0.030	-0.016	-0.003

the predicted M1 transitions between the vector and pseudoscalar charmonia appear to have significant discrepancies. Namely, the predicted partial decay width  $\Gamma^{NR}(J/\psi \rightarrow \gamma\eta_c) \simeq 2.4 \sim 2.9$  keV [53, 54] is obviously larger than the experimental data in PDG2010 [39],  $Br(J/\psi \rightarrow \gamma\eta_c) = (1.7 \pm 0.4)\%$ , i.e.,  $\Gamma(J/\psi \rightarrow \gamma\eta_c) = (1.58 \pm 0.37)$  keV. The PDG2010 value is mainly weighted by the CLEO data  $Br(J/\psi \rightarrow \gamma\eta_c) = (1.98 \pm 0.09 \pm 0.30)\%$  [55]. Although the CLEO data bring the experimental and theoretical values closer, the discrepancy remains nontrivial after taking into account another channel  $\psi' \rightarrow \gamma\eta_c$ : the potential models predicted  $\Gamma^{NR}(\psi' \rightarrow \gamma\eta_c) \simeq 9.6 \sim 9.7$  keV, i.e.,  $Br(\psi' \rightarrow \gamma\eta_c) \simeq 3.2\%$ , while the CLEO measurement gives  $Br(\psi' \rightarrow \gamma\eta_c) = (4.32 \pm 0.16 \pm 0.60) \times 10^{-3}$  [55].

Theoretical efforts of studying the charmonium EM M1 transitions in the framework of nonrelativistic multipole expansions can be found in [56–63]. Recently, a nonrelativistic effective field theory was applied to  $J/\psi \rightarrow \gamma\eta_c$  [62], in which the radiative decay width  $\Gamma(J/\psi \rightarrow \gamma\eta_c) = (1.5 \pm 1.0)$  keV up to correction of  $O(v_c^2/m_c^2)$  was obtained with a rather large uncertainty. This approach becomes unreliable in  $\psi' \rightarrow \gamma\eta_c$ , because the  $c\bar{c}$  pair cannot be treated as a weakly bound system anymore. The lattice QCD calculations of these processes were reported in [64, 65]. In the quenched approximation the result for  $J/\psi \rightarrow \gamma\eta_c$  turns out to be in agreement with the potential models, while that for  $\psi' \rightarrow \gamma\eta_c$  is much smaller and compatible with the data within uncertainties. That is, the quenched lattice QCD does not resolve the puzzle completely.

A possible resolution arises from the accommodation of open threshold effects as an unquenched mechanism in the charmonium M1 transitions [32, 33]. Because of the presence of the open  $D\bar{D}$  threshold, the transition  $\psi' \rightarrow \gamma\eta_c$  would experience more influence from the  $D\bar{D}$  threshold, which naturally lowers the partial decay width predicted by the potential models. In contrast, the open threshold effects on  $J/\psi \rightarrow \gamma\eta_c$  are relatively small, since the mass of  $J/\psi$  is located rather far away from the  $D\bar{D}$  threshold. However, the uncertainties from the unquenched effects are significant as shown in [32, 33], so there is still room for the glueball- $\eta_c$  mixing mechanism in the present experimental and theoretical situations.

Starting with Eq. (28), we express the quenched M1 transition amplitude as

$$\begin{aligned} \langle \eta_c | H_{em} | J/\psi, \psi' \rangle &= -s\phi_Q \langle g | H_{em} | J/\psi, \psi' \rangle + c\phi_Q \langle \eta_Q | H_{em} | J/\psi, \psi' \rangle \\ &\simeq (-s\phi_Q \frac{\alpha_s}{\pi} + c\phi_Q) \langle \eta_Q | H_{em} | J/\psi, \psi' \rangle, \end{aligned} \quad (36)$$

where  $\langle \eta_Q | H_{em} | J/\psi, \psi' \rangle$  is equivalent to the potential-model M1 transition amplitude for the  $J/\psi, \psi'$  mesons, and the gluon counting rule has been implemented in the second line. One immediately notices that in order to lower the M1 transition partial width, a positive  $\phi_Q$  is favored. Given  $\phi_Q \simeq 11^\circ$ , the quenched M1 transition partial widths are lowered by about 7%. That is, the glueball- $\eta_c$  mixing does improve the overall consistency between the potential-model predictions and the data for the charmonium M1 transitions.

#### D. $B^0 \rightarrow \eta^{(\prime)} K_S$ decays

As stated in the Introduction, a potential deviation has been detected between the mixing-induced CP asymmetries in the tree-dominated decays  $B \rightarrow J/\psi K_S$  and in the penguin-dominated decays  $B \rightarrow \eta' K_S$ , which demands a deeper theoretical understanding. Besides, the branching ratios of the  $B \rightarrow \eta' K$  decays predicted in the PQCD approach up to NLO are still much lower than the data. The value of  $f_{\eta'}^c$  obtained in Sec. III A suggests a quantitative reexamination of the tree contribution from  $B \rightarrow \eta_c K$  to the direct and mixing-induced CP asymmetries and the branching ratios of the  $B^0 \rightarrow \eta^{(\prime)} K_S$  decays. The  $\lambda_{CP}$  factor for this study is defined as

$$\lambda_{CP} = \eta_f e^{-2i\beta} \frac{\langle f | H_{eff} | \bar{B}^0 \rangle}{\langle f | H_{eff} | B^0 \rangle}, \quad (37)$$

with the eigenvalue  $\eta_f = -1$  and the weak phase  $\beta$ . The direct and mixing-induced CP asymmetries are then derived from

$$A_{CP}^{dir} = \frac{|\lambda|^2 - 1}{1 + |\lambda|^2}, \quad A_{CP}^{mix} = \frac{2Im\lambda}{1 + |\lambda|^2}. \quad (38)$$

For the  $B^0 \rightarrow \eta' K_S$  decays without the  $\eta_c$  mixing, one has

$$\lambda_{CP} = \eta_f e^{-2i\beta} \frac{V_{ub}V_{us}^*T_{\eta'K} - V_{tb}V_{ts}^*P_{\eta'K}}{V_{ub}^*V_{us}T_{\eta'K} - V_{tb}^*V_{ts}P_{\eta'K}}, \quad (39)$$

where  $V$ 's are the CKM matrix elements, and  $T$  and  $P$  represent the tree and penguin amplitudes, respectively. These decay amplitudes have been evaluated up to NLO in the PQCD approach [34], which lead to  $A_{CP}^{dir} = 0.024$  and  $A_{CP}^{mix} = 0.667^1$ . The tree amplitude  $T_{\eta_c K}$  has been also calculated in NLO PQCD [35], which gives the branching ratio  $Br(B^0 \rightarrow \eta_c K^0) = 5.5 \times 10^{-4}$ . Because the  $B$  meson decay constants  $f_B = 0.21$  GeV and  $f_B = 0.19$  GeV were adopted in [34] and [35], respectively, and the  $\eta_c$  meson decay constants  $f_{\eta_c} = 0.478$  GeV were derived in Eq. (34), and  $f_{\eta_c} = 0.42$  GeV was adopted in [35], we multiply  $T_{\eta_c K}$  in [35] by a factor  $(0.21/0.19)(0.478/0.42) = 1.26$  for consistency. The increased NLO PQCD prediction  $Br(B^0 \rightarrow \eta_c K^0) = 8.7 \times 10^{-4}$  then agrees well with the data  $(8.7 \pm 1.9) \times 10^{-4}$  [52]. Note that the relative strong phases among the above amplitudes  $T_{\eta'K}$ ,  $P_{\eta'K}$  and  $T_{\eta_c K}$  are known in the PQCD approach, so we do not encounter the difficulty mentioned in [22], and can derive the modified CP asymmetries without ambiguity. The inclusion of the  $B \rightarrow \eta_c K$  channel into Eq. (39),

$$\lambda_{CP} = \eta_f e^{-i2\beta} \frac{V_{ub}V_{us}^*T_{\eta'K} - V_{tb}V_{ts}^*P_{\eta'K} + c\theta s\phi_G s\phi_Q V_{cb}V_{cs}^*T_{\eta_c K}}{V_{ub}^*V_{us}T_{\eta'K} - V_{tb}^*V_{ts}P_{\eta'K} + c\theta s\phi_G s\phi_Q V_{cb}^*V_{cs}T_{\eta_c K}}, \quad (40)$$

yields  $A_{CP}^{dir} = 0.023$  and  $A_{CP}^{mix} = 0.664$ . It is seen that the  $\eta_c$  mixing causes a negligible effect on the CP asymmetries with  $A_{CP}^{mix}$  moving slightly toward the central value of the data,  $0.59 \pm 0.07$  [52]. The result of  $A_{CP}^{dir}$  is consistent with the data  $0.05 \pm 0.05$  [52]. Nevertheless, the  $\eta_c$  mixing brings the branching ratio  $Br(B^0 \rightarrow \eta' K^0) = 50 \times 10^{-6}$  in NLO PQCD [35] to  $59 \times 10^{-6}$ , which becomes closer to the data  $(66.1 \pm 3.1) \times 10^{-6}$  [52]. Note that the enhancement of the above branching ratio due to the charm content of the  $\eta'$  meson is larger than few percents estimated in [66].

Similarly, we investigate the impact of the  $\eta_c$  mixing on the  $B^0 \rightarrow \eta K_S$  decays. Without the charm content of the  $\eta$  meson, the NLO PQCD analysis gave  $A_{CP}^{dir} = -0.128$  and  $A_{CP}^{mix} = 0.659$  [34]. The  $\eta_c$  mixing then modifies the above values into  $A_{CP}^{dir} = -0.123$  and  $A_{CP}^{mix} = 0.644$ , namely, with a negligible effect. The branching ratio  $Br(B^0 \rightarrow \eta K^0)$ , becoming  $2.3 \times 10^{-6}$  from  $2.1 \times 10^{-6}$ , is almost not changed by the  $\eta_c$  mixing. The result is a bit higher than the data  $Br(B^0 \rightarrow \eta K^0) = (1.12_{-0.28}^{+0.30}) \times 10^{-6}$  [52]. However, if using  $\theta = -11^\circ$  in the present work, the destructive interference for the  $B \rightarrow \eta K$  decays from the  $\eta$ - $\eta'$  mixing would be stronger, which will lower their branching ratios.

#### IV. SUMMARY

In this paper we have extended the  $\eta$ - $\eta'$ - $G$  mixing formalism constructed in our previous work to accommodate the  $\eta_c$  meson in a tetramixing scheme. The additional mixing angle between  $G$  and  $\eta_c$  was determined to be about  $11^\circ$  from the observed widths of the  $\eta_c$  meson decays into light hadrons and  $\gamma\gamma$ . This mixing would have a leading impact on the  $\eta_c \rightarrow g\gamma$  width instead of on the  $\eta_c \rightarrow \gamma\gamma$  one, such that the  $O(\alpha_s v^2)$  corrections in NRQCD [43] can be parametrized out. More precise measurement of the  $\eta_c$  total decay width and its decays into  $\gamma\gamma$  can provide better constraints on the mixing scheme. Our tetramixing matrix was found to be consistent with that constructed from the  $SO(3) \otimes SO(3)$  parametrization with a fit to data of relevant transition form factors. Contrary to general opinions in the literature, the present work suggests an reexamination of effects from the charm content of the  $\eta^{(\prime)}$  meson on the  $B \rightarrow \eta^{(\prime)} K$  decays.

We have shown that such a tetramixing scheme does increase the pseudoscalar density  $m_{qq}$  to above the pion mass, which thus enhances theoretical predictions for the  $B \rightarrow \eta'$  transition form factors and the  $B \rightarrow \eta' K$  branching ratios [28]. It has been observed that the charm content of the  $\eta'$  meson provides 18% enhancement of the  $B \rightarrow \eta' K$  branching ratios. The combined mechanisms can push the predicted values in NLO PQCD to the data easily, so the puzzle due to the large  $Br(B \rightarrow \eta' K)$  is resolved. With this work, we postulate that the charm content of the  $\eta'$  meson plays a more important role than the gluonic content in accommodating the large  $Br(B \rightarrow \eta' K)$ . Nevertheless, the  $\eta_c$  mixing has negligible effects on the direct and mixing-induced CP asymmetries of the  $B \rightarrow \eta^{(\prime)} K_S$  decays, though the mixing-induced CP asymmetries move slightly toward the central value of the data.

We have also investigated the impact of the tetramixing on the present theoretical and experimental observations of charmonium magnetic dipole transitions, and similar improvement is also seen: the gluonic content of the  $\eta_c$  meson decreases the decay widths  $\Gamma(J/\psi, \psi' \rightarrow \gamma\eta_c)$  predicted by the nonrelativistic potential models by 7%. It has been confirmed that the  $\eta_c$  mixing does not modify the prediction in [25] for the pseudoscalar glueball mass in the

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<sup>1</sup> The quoted value of  $A_{CP}^{mix}$  differs from that presented in [34], since the input of the weak phase  $\beta$  has been corrected.

vicinity of 1.4-1.5 GeV. This result makes the  $\eta(1405)$  meson an interesting candidate for the pseudoscalar glueball. However, one should be aware of the complexity of underlying dynamics, such as the octet-glueball coupling [51] and intermediate meson rescattering in this mass region [67], which certainly affect the determination of the pseudoscalar glueball mass.

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